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A NON-LINEAR RELAY MODEL FOR

POST-STALL OSCILLATIONS

by

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ABSTRACT:

A simplified non-linear relay model is developed to describe observed post-stall oscillations in aircraft. The predictions of the model are evaluated against results obtained by numerical techniques, and shown to yield close agreement.

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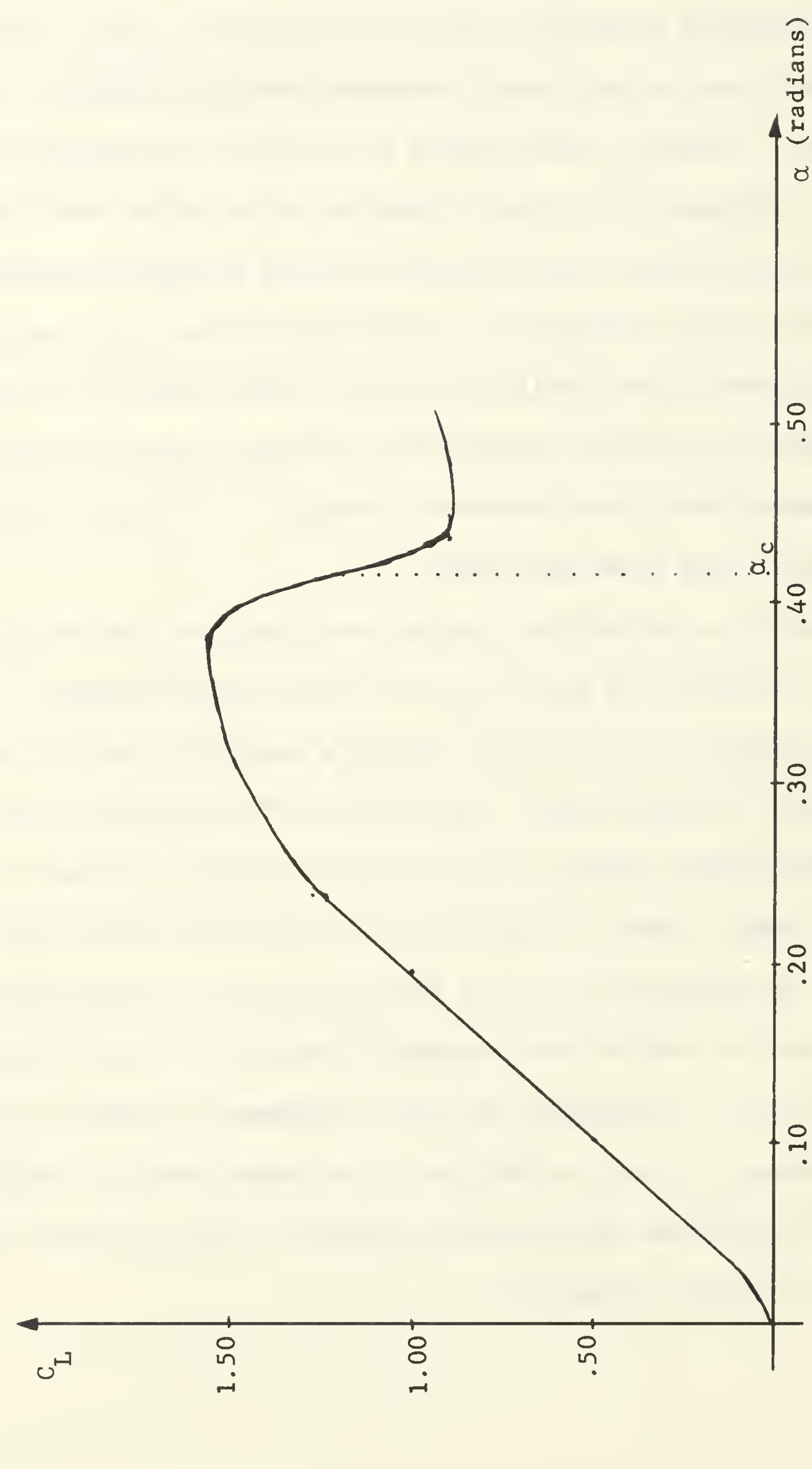
1. INTRODUCTION

Considerable literature exists describing the static characteristics of aircraft near, or at, stall, including detailed analyses of stall tendencies. However, there appears to be little information on, or analytic treatment of, aircraft dynamics in the region around stall, and especially post-stall time histories in which divergent oscillations arise. The latter phenomenon is quite interesting, since experienced pilots apparently are familiar with the "rocking chair" or "porpoising" stall, where the aircraft, while under constant elevator control, alternately enters stall, then recovers, leading to a vertical oscillation of the aircraft nose about some point.

Normally we define the airplane stall angle as that angle where the vehicle's trimmed lift curve (C_L vs α) has a global maximum. It is well known that the lift curve then exhibits a negative slope for some range of α above the stall angle. Therefore, we shall also use here α_c, the stall break angle, defined as that angle where the lift curve has maximum negative slope. There is a paucity of information on the nature of the lift curve in the post-stall or stall break region, since experimental wind tunnel data has usually been smoothed through this region based on engineering intuition. In addition, an actual treatment to define the stall break for a physical aircraft is difficult since model data are subject to scaling difficulties and aircraft parameter in the non-linear region is limited at present. (Figure 1)



Typical C_L vs α Curve.

Figure 1

In his thesis, Frederiksen [1] considered the problem of divergent oscillations and limit cycles in this region. His linear analysis was able to show, numerically for a single aircraft, that a sufficiently large negative slope in $C_{L\alpha}$ in this region led to roots for the characteristic equation which had positive real parts (i.e. exponentially growing). This behavior is, of course, transitory, and disappears by the time the aircraft returns to a configuration where the lift slope is positive. However, it did provide an insight into the mechanism that might cause these oscillations. Furthermore, he was able to simulate numerically time histories in which a "rocking chair" stall did occur. However, due to the numerical nature of his work and his concentration on a single aircraft, his results are not readily extendible, or suitable for general analysis.

This investigation attempts to analytically model the post-stall behavior in a general class of aircraft. Our main hypothesis is that, if the slope of the lift curve is sufficiently negative in the region beyond the stall break, then the aircraft will behave as if it sees a sharp, essentially instantaneous loss of lift. If this is the case, then the lift curve can be more conveniently modeled in this region by the introduction of a hypothetical "relay" acting at the stall break angle. We assume that, except for this nonlinearity, C_L, C_M and C_D can be adequately approximated by linear functions. Those assumptions, together with conditions on the magnitude of various coefficients, lead to a system of coupled, constant coefficient, first order differential equations, with a forcing term which has a jump discontinuity across a line in the phase plane. Analysis of this system yields a necessary and sufficient condition for the existence of a limit cycle, and transcendental equations predicting the period of this cycle. These conditions are then applied to the case considered by Frederiksen, and it is shown our model yields excellent agreement with his results.

2. DERIVATION OF THE MODEL

The aircraft, assumed to be in longitudinal flight, will be described by means of the standard coordinate system shown in figure 2. In this system, the equations of motion become

$$\dot{u} = \frac{\{ T + qS [C_L \sin\alpha - C_D \cos\alpha]\}}{m} - g \sin\theta - w\,\dot{\theta} \quad , \tag{1}$$

$$\dot{w} = - \frac{qS [C_L \cos\alpha + C_D \sin\alpha]}{m} + g \cos\theta + u\,\dot{\theta} \quad , \tag{2}$$

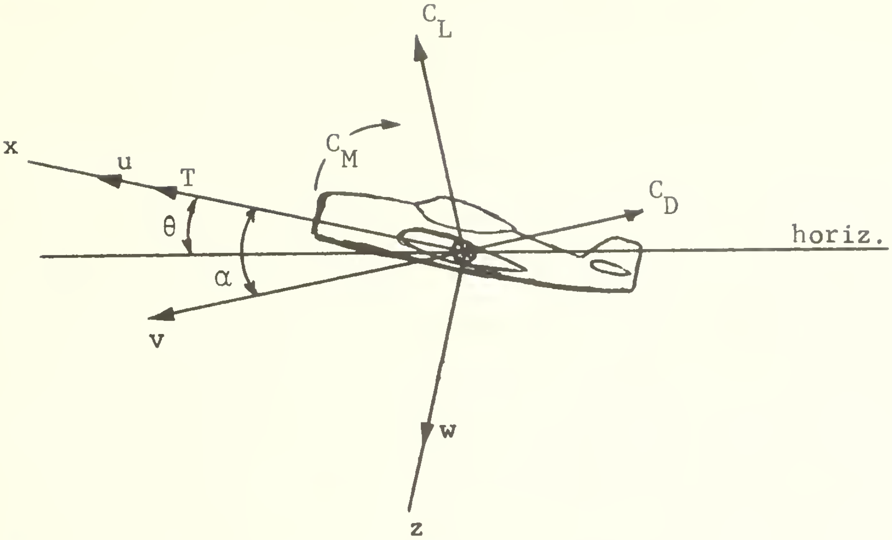
$$\ddot{\theta} = \frac{qS\bar{c}}{I_y} C_M \quad , \tag{3}$$

and,

$$\dot{\alpha} = \frac{\{ - \dot{u} \sin\alpha + \dot{w} \cos\alpha\}}{v} \tag{4}$$

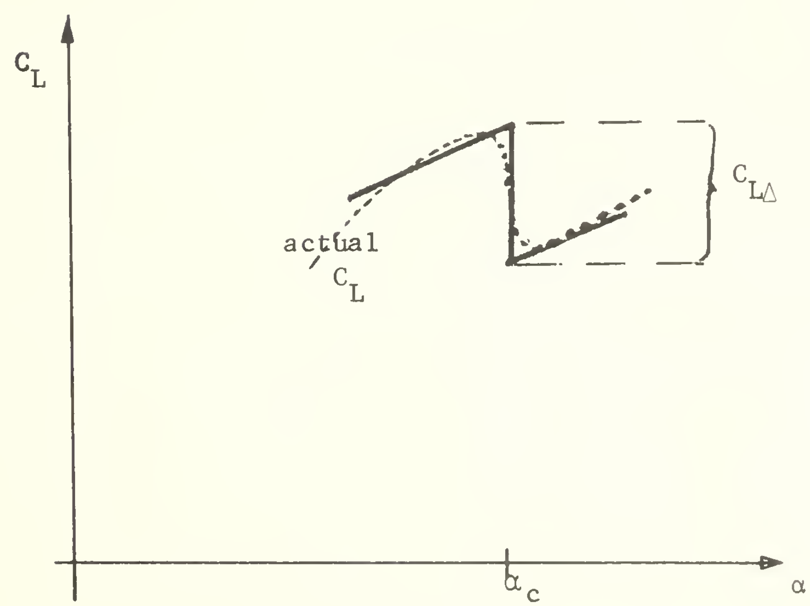
(The terms in (1) - (4) are defined in the Appendix.)

These equations are obviously non-linear, even though the coefficients may be linear or constant. In addition, though, in general, over wide ranges of variation, the coefficients are non-linear functions of the variables. In most applications where the range of the variables is



Aircraft Coordinate System

Figure 2



Approximation to C_L

Figure 3

small, they may, however, usually be adequately approximated by linear functions. As is obvious from figure 1, this is not the case though for C_L in the neighborhood of the stall break.

We shall proceed under the assumption that any motion of the aircraft takes place in a small region about the stall break angle. Furthermore we shall assume that C_M and C_D can still be approximated linearly in this region. We suppose the aircraft is trimmed for constant attitude and angle of attack, denoted by θ_o and α_o respectively, and that the elevator angle and thrust, denoted by δ_e and T, are constant. The velocity of the aircraft in the trim configuration is denoted by v_o. Frederiksen's results do not show a significant variation in v, compared to that in α and θ during limit cycle behavior, so we assume

$$v = v_o .$$

Considering now perturbations off the trim configuration, we let

$$\Delta\alpha = \alpha - \alpha_o \ , \quad \text{and} \quad \Delta\theta = \theta - \theta_o$$

As noted above, we will approximate C_M and C_D linearly, so that

$$C_D = C_{DO} + C_{D\alpha}\,\Delta\alpha \ , \tag{5}$$

and,

$$C_M = - C_{MO} - C_{M\alpha}\,\Delta\alpha - \frac{\bar{c}}{2v_o}\, C_{M\dot{\theta}}\,\Delta\dot{\theta} - \frac{\bar{c}}{2v_o}\, C_{M\dot{\alpha}}\,\Delta\dot{\alpha} - C_{M\delta_e}\,\delta_e \ . \tag{6}$$

The key ingrediant of this investigation is modeling C_L at the stall break. Reference to figure 1 indicates that, if the negative slope in the region past the stall break is sufficiently steep, then an aircraft entering this region experiences an almost instantaneous loss in lift. This is essentially equivalent to the action of a hypothetical "relay" in the lift curve function C_L. Thus, we shall model the lift curve as:

$$C_L = C_{L0} + C_{L\delta_e} \delta_e + \frac{\bar{c}}{2v_o} C_{L\dot{\theta}} \Delta\dot{\theta}$$

$$+ C_{L\alpha_c} \Delta\alpha - C_{L\Delta} \operatorname{sgn}(\alpha - \alpha_c) \quad , \qquad (7)$$

where $\operatorname{sgn}(x) = |x|/x$, $x \neq 0$, $\operatorname{sgn}(0) = 0$, and α_c denotes the stall break angle. Figure 3 shows a sketch of a C_L curve generated by (7) superimposed on the C_L curve from figure 1. (Note that, to facilitate analysis, we have assigned algebraic signs so that, in normal cases, all the coefficients are positive.)

Note now that $\dot{u}$ and $\dot{w}$ which appear in (4) can be replaced by their equivalent expression in terms of (1) and (2). If we do this, we find (since $u = v \cos \alpha$, and $w = v \sin \alpha$):

$$\dot{\alpha} = -\frac{T}{mv} \sin \alpha - \frac{qS}{mv} C_L + \frac{g}{v} \cos(\theta - \alpha) + \dot{\theta} \qquad (8)$$

But now, using our assumption on v, and the fact that

$$\dot{\alpha} = \Delta\dot{\alpha} \ , \quad \text{and} \quad \dot{\theta} = \Delta\dot{\theta} \ , \text{ and } \ddot{\theta} = \Delta\ddot{\theta} \ ,$$

we have,

$$\Delta\dot{\alpha} = -\frac{T}{mv_o} \sin(\alpha_o + \Delta\alpha) - \frac{qS}{mv_o} C_L + \frac{g}{v_o} \cos(\theta_o - \alpha_o + \Delta\theta - \Delta\alpha)$$

$$+ \Delta\dot{\theta} \ , \qquad (9)$$

and,

$$\ddot{\Delta\theta} = \frac{q S \bar{c}}{I_y} C_M \quad . \tag{10}$$

But then, according to the assumed forms we have for C_L and C_M, we see that $\dot{\Delta\alpha}$ and $\ddot{\Delta\theta}$ depend only on $\Delta\alpha$, $\Delta\theta$, and $\dot{\Delta\theta}$ (since (9) can be used to eliminate the $\dot{\Delta\alpha}$ term from C_M.) Thus, for the remainder of our investigation, we shall be concerned only with equations (6), (7), (9), and (10).

Now, under the assumption that $\Delta\alpha$ and $\Delta\theta$ are small, we see:

$$\sin(\alpha_o + \Delta\alpha) \doteq \sin \alpha_o + (\cos \alpha_o)\, \Delta\alpha \quad ,$$

and,

$$\cos(\theta_o - \alpha_o + \Delta\theta - \Delta\alpha) \doteq \cos(\theta_o - \alpha_o) - \sin(\theta_o - \alpha_o)\, \Delta\theta$$

$$+ \sin(\theta_o - \alpha_o)\, \Delta\alpha \quad ,$$

where second order and higher terms have been neglected. Using these relations, and the defining equation for C_L, we can reduce (9) to the following:

$$\dot{\Delta\alpha} = \{-\frac{T}{mv_o} \sin \alpha_o - \frac{qS}{mv_o} [C_{L0} + C_{L\delta_e} \delta_e] + \frac{g}{v_o} \cos(\theta_o - \alpha_o)\}$$

$$+ \{ \frac{g}{v_o} \sin(\theta_o - \alpha_o) - \frac{T}{mv_o} \cos \alpha_o - \frac{qS}{mv_o} C_{L\alpha_c} \}\, \Delta\alpha$$

$$- \{ \frac{g}{v_o} \sin(\theta_o - \alpha_o)\}\Delta\theta + \{1 - \frac{qS\bar{c}}{2mv_o^2} C_{L\dot{\theta}} \}\, \dot{\Delta\theta} + \frac{qS}{mv_o} C_{L\Delta}\, \mathrm{sgn}(\Delta\alpha - (\alpha_c - \alpha_o)). \tag{11}$$

Now, using this expression to replace $\Delta\dot{\alpha}$ in (7), the equation for C_M, and substituting the resulting expression into (10), yields:

$$\Delta\ddot{\theta} = \frac{qS\bar{c}}{I_y}\Bigg[\{- C_{M0} - C_{M\delta_e}\delta_e - \frac{\bar{c}}{2v_o} C_{M\dot{\alpha}} [- \frac{T}{mv_o} \sin \alpha_o - \frac{qS}{mv_o} (C_{L0} + C_{L\delta_e}\delta_e) - \frac{g}{v_o} \cos(\theta_o - \alpha_o)]\}$$

$$- \{C_{M\alpha} + \frac{\bar{c}}{2v_o} C_{M\dot{\alpha}} [\frac{g}{v_o} \sin(\theta_o - \alpha_o) - \frac{T}{mv_o} \cos \alpha_o - \frac{qS}{mv_o} C_{L\alpha_c}]\} \Delta\alpha$$

$$+ \{\frac{g\bar{c}}{2v_o^2} C_{M\dot{\alpha}} \sin(\theta_o - \alpha_o)\} \Delta\theta - \frac{\bar{c}}{2v_o} \{C_{M\dot{\theta}} + C_{M\dot{\alpha}} [1 - \frac{qS\bar{c}}{2mv_o^2} C_{L\dot{\theta}}]\} \Delta\dot{\theta}$$

$$- \frac{qS\bar{c}}{2mv_o^2} C_{M\dot{\alpha}} C_{L\Delta} \operatorname{sgn}(\Delta\alpha - (\alpha_c - \alpha_o))\Bigg] \quad . \qquad (12)$$

Equations (11) and (12) then describe the model we shall analyze.

In practice, many of the terms which appear in (11) and (12) are negligible. Calculations, using representative values, will show that for many, if not all cases, little is lost by assuming:

$$\left| \frac{qS\bar{c}}{2mv_o^2} C_{L\dot{\theta}} \right| << 1 , \qquad \left| \frac{\bar{c}}{2v_o} C_{M\dot{\alpha}} \right| << 1 ,$$

and also that,

$$\left| \frac{g}{v_o} \sin(\theta_o - \alpha_o) - \frac{T}{mv_o} \cos \alpha_o - \frac{qS}{mv_o} C_{L\alpha_c} \right| << 1 ,$$

and,

$$\left| \frac{g}{v_o} \sin(\theta_o - \alpha_o) \right| << 1 .$$

Also, we have not yet used our assumption that the aircraft was trimmed for flight at α_o, θ_o. This assumption says that when

$$\Delta\alpha = \Delta\theta = \Delta\dot{\theta} = 0 ,$$

then,

$$\Delta\alpha = \Delta\ddot{\theta} = 0.$$

Thus, imposition of the trim conditions reduces (11) and (12) respectively to:

$$0 = \{-\frac{T}{mv_o}\sin\alpha_o - \frac{qS}{mv_o}[C_{L0} + C_{L\delta_e}\delta_e] + \frac{g}{v_o}\cos(\theta_o - \alpha_o)\}$$

$$- \frac{qS}{mv_o} C_{L\Delta}\,\mathrm{sgn}(\alpha_c - \alpha_o) \quad , \tag{13}$$

and,

$$0 = - C_{M0} - C_{M\delta_e}\delta_e \quad . \tag{14}$$

If we now incorporate (13), (14), and our assumptions about small terms into (11) and (12), we arrive at

$$\Delta\dot{\alpha} = \Delta\dot{\theta} + \frac{qS}{mv_o} C_{L\Delta}\{\mathrm{sgn}(\Delta\alpha - (\alpha_c - \alpha_o)) + \mathrm{sgn}(\alpha_c - \alpha_o)\} , \tag{15}$$

and,

$$\Delta\ddot{\theta} = -\frac{qS\bar{c}}{I_y} C_{M\alpha}\Delta\alpha - \frac{qS\bar{c}^2}{2v_oI_y}\{C_{M\dot{\theta}} + C_{M\dot{\alpha}}\}\Delta\dot{\theta}$$

$$- \frac{(qS\bar{c})^2}{2mI_yv_o^2} C_{M\dot{\alpha}} C_{L\Delta}\{\mathrm{sgn}(\Delta\alpha - (\alpha_c - \alpha_o) + \mathrm{sgn}(\alpha_c - \alpha_o)\} . \tag{16}$$

We now define the quantities β, ω, a, and ζ by:

$$\frac{qS\bar{c}}{I_y} C_{M\alpha} = (\beta^2 + \omega^2) > 0 \quad ,$$

$$\frac{qS\bar{c}^2}{2v_o I_y} \{C_{M\dot{\theta}} + C_{M\dot{\alpha}}\} = 2\beta > 0 \tag{17}$$

$$\frac{qS}{mv_o} C_{L\Delta} = a > 0 \ ,$$

and,

$$\frac{qS\bar{c}^2}{2v_o I_y} C_{M\dot{\alpha}} = \zeta > 0 \quad .$$

Since it becomes important in our later discussion, note here that the quantity:

$$(2\beta - \zeta) = \frac{qS\bar{c}^2}{2v_o I_y} C_{M\dot{\theta}} > 0 \quad .$$

Thus our model for the flight dynamics of an aircraft, trimmed for longitudinal flight at an angle of attack near the stall break angle, becomes:

$$\Delta\dot{\alpha} = \Delta\dot{\theta} + a\{sgn(\Delta\alpha-(\alpha_c - \alpha_o)) + sgn(\alpha_c-\alpha_o)\} \ . \tag{18}$$

$$\Delta\ddot{\theta} = -(\beta^2 + \omega^2)\Delta\alpha - 2\beta\Delta\dot{\theta} - \zeta a\{sgn(\Delta\alpha-(\alpha_c - \alpha_o)) + sgn(\alpha_c-\alpha_o)\}. \tag{19}$$

(Note that (18) and (19) describe a system which is second order in $\Delta\alpha$ and $\Delta\dot{\theta}$. Since the original system (1) - (4) was equivalent to a fourth order system in α, θ, $\dot{\theta}$, and v, our model will obviously not describe the full range of dynamics. However, we will show that it does appear to predict the known limit cycle behavior. In essence, what we are doing in equations (18) and (19) is basing our analysis on the influence of the stall break on the short period mode of the aircraft, and neglecting the long period mode. Furthermore, the mathematical advantages of dealing with a second order system are enormous.)

3. FLIGHT WITH TRIM AT THE STALL BREAK ANGLE

We begin our investigation of the model described by (18) and (19) with the case $\alpha_c = \alpha_o$, that is the aircraft is trimmed for flight at the stall break. Then since $\alpha_c - \alpha_o = 0$, the second sgn function in both (18) and (19) disappears. If we let $x_1 = \Delta\dot{\theta}$ and $x_2 = \Delta\alpha$, our system reduces to:

$$\dot{x}_1 = -2\beta x_1 - (\beta^2 + \omega^2) x_2 - \zeta a \operatorname{sgn} x_2$$

$$\dot{x}_2 = x_1 + a \operatorname{sgn} x_2 \tag{20}$$

<u>A. The Phase Plane</u>. Consider the analytic behavior of this system in the (x_1 , x_2) phase plane. The line $x_2 = 0$ divides the plane into two regions, where separately the equations are linear, constant coefficient and non-homogeneous. For example, in $x_2 > 0$, the system is:

$$\dot{x}_1 = -2\beta x_1 - (\beta^2 + \omega^2)x_2 - \zeta a \quad ,$$

$$\dot{x}_2 = x_1 + a \quad . \tag{21}$$

A key observation now is that (20) is symmetric, i.e.

$$\{ (x_1(t), x_2(t)) \mid \tau_1 \le t \le \tau_2 \}$$

represents an arc of a trajectory if and only if

$$\{ (-x_1(t), -x_2(t)) \mid \tau_1 \le t \le \tau_2 \}$$

also does. Thus we need direct our attention solely to the region $x_2 > 0$, and draw any needed information about $x_2 < 0$ from symmetry arguments. We shall use the usual convention in referring to the line $x_2 = 0$ as the switching line.

We start by considering the critical points of the system, i.e., the points where $\dot{x}_1 = \dot{x}_2 = 0$. These are given by:

$$x_1 = x_2 = 0 \quad ,$$

$$x_1 = a, \qquad x_2 = -\frac{(2\beta - \zeta)}{(\beta^2 + \omega^2)} a \quad ,$$

and,

$$x_1 = -a, \qquad x_2 = \frac{(2\beta - \zeta)}{(\beta^2 + \omega^2)} a \quad .$$

(These critical points actually occur, since, as we have noted, $(2\beta - \zeta) > 0$). The presence of the sgn term in (20) implies that the critical point at (0,0) will be unstable. However, it is easily shown that the homogeneous part of (21) has the characteristic polynomial

$$\lambda^2 + 2\beta\lambda + (\beta^2 + \omega^2) \quad ,$$

whose roots must have negative real parts according to our definition of β and ω in (17). Hence, the two critical points not at the origin are locally asymptotically stable. (That is, small perturbations off this point decay back into it.) We shall show shortly these are not globally asymptotically stable.

<u>B. Conditions for a Limit Cycle</u>. We now consider the conditions under which (20) can have a limit cycle (i.e. periodic) solution. Obviously from the stability discussion above, any limit cycle trajectory must pass through the switching line. Furthermore, it must by symmetric with respect to the origin. Thus, if we consider the arc of the limit cycle in the region $x_2 > 0$, and assume this arc originates on the switching line at the point $(x_1^*, 0)$, then, after a half-period, it must terminate at the point $(-x_1^*, 0)$. Thus, if we denote the half-period by T, we have a necessary condition for a limit cycle of period $2T$ is:

$$\begin{aligned} x_1(0) &= - x_1(T) \quad , \\ x_2(0) &= x_2(T) = 0 \quad . \end{aligned} \tag{22}$$

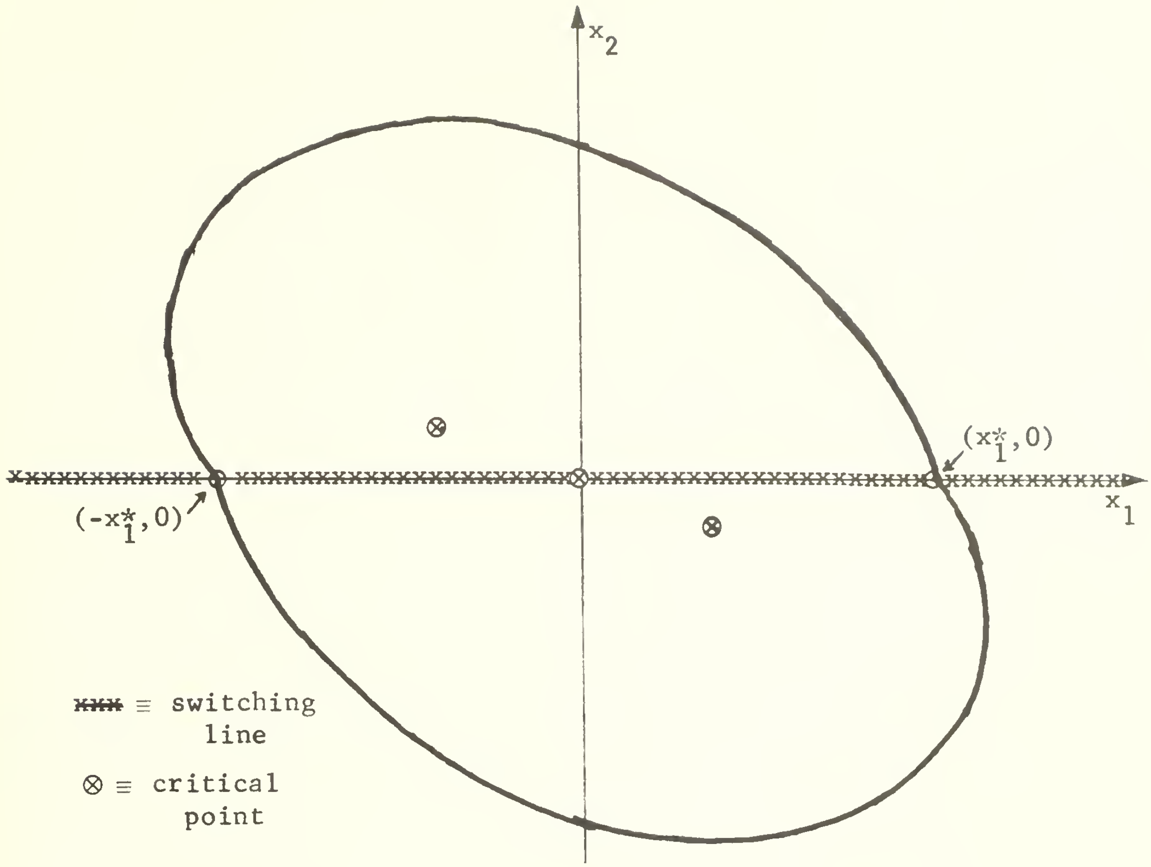
(This situation is represented in figure 4.)

It is easily shown that in the region $x_2 > 0$ the solution to (20) is given by:

$$\begin{aligned} x_1(t) = -c_1 \, e^{-\beta t}(\beta \cos \omega t + \omega \sin \omega t) \\ + c_2 \, e^{-\beta t}(\omega \cos \omega t - \beta \sin \omega t) - a \quad , \end{aligned} \tag{23}$$

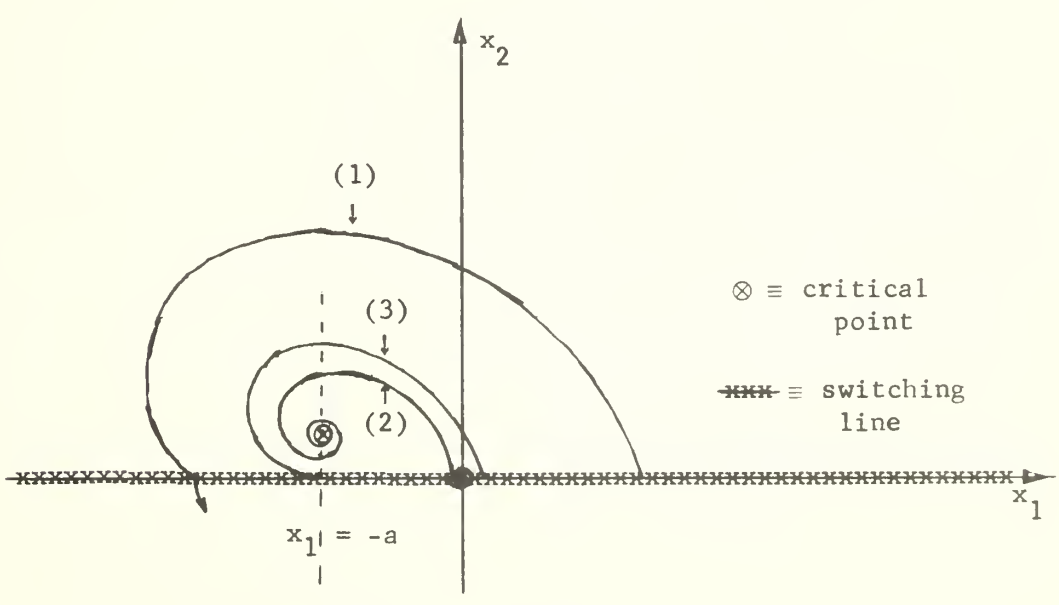
and,

$$x_2(t) = c_1 \, e^{-\beta t} \cos \omega t + c_2 \, e^{-\beta t} \sin \omega t + \frac{(2\beta - \zeta)}{(\beta^2 + \omega^2)} \, a \; . \tag{24}$$



The Limit Cycle Necessary Condition

Figure 4



The $x_2 > 0$ Region

Figure 5

(These simply represent arcs of counterclockwise spirals centered at the point $(-a, \frac{(2\beta - \zeta)}{(\beta^2 + \omega^2)} a)$, which is the critical point in the upper half plane. Thus every curve originating on the switching line must follow a trajectory that carries it above this point. (A typical such curve is labeled (1) in figure 5.) Furthermore, since the critical point (if it exists) is asymptotically stable, there must be some curves entering the region which never exit it, but decay down to the critical point. (One of these is shown as (2) in figure 5.) Finally, since any trajectory intersecting the line $x_1 = -a$ has a horizontal slope at the point of intersection (see eqn (21)), then the boundary between the decaying curves and those that exit to $x_2 < 0$ must just be that curve which terminates at $(-a,0)$. (This curve is labeled (3) in figure 5.)

But also note that the time interval represented by a trajectory in the region $x_2 > 0$ is a function of the counterclockwise angle (as seen from $(-a, \frac{(2\beta - \zeta)}{(\beta^2 + \omega^2)} a)$) between the initial point $(x_1(0), 0)$, and the point at which the trajectory first touches the switching line again. Thus, any curve which represents a half-period of a limit cycle must spend less time in $x_2 > 0$ than T_o, where T_o is the time represented by the curve (3) in figure 5. Thus T satisfies:

$$T < T_o ,$$

where,

$$x_1(T_o) = -a$$

$$x_2(0) = x_2(T_o) = 0 . \tag{25}$$

Conditions (22) and (25) can now be seen to provide both the necessary and sufficient conditions for existence of a limit cycle of period $2T$.

The condition $x_2(0) = x_2(T) = 0$, applied to (24), immediately leads to:

$$c_1 = - \frac{(2\beta - \zeta)}{(\beta^2 + \omega^2)} a \quad , \tag{26}$$

and,

$$c_1(e^{\beta T} - \cos \omega T) = c_2 \sin \omega T \ . \tag{27}$$

Notice, according to these, that $T \neq \frac{\pi}{\omega}$.

The second condition, on $x_1(t)$, when applied to (23), implies:

$$\begin{aligned} x_1(0) = - \beta c_1 + \omega c_2 - a &= - x_1(T) \\ &= c_1 e^{-\beta T}(\beta \cos \omega T + \omega \sin \omega T) \\ &\quad - c_2 e^{-\beta T}(\beta \cos \omega T - \omega \sin \omega T) + a \end{aligned}$$

or,

$$\begin{aligned} &c_1(\beta + \beta e^{-\beta T} \cos \omega T + \omega e^{-\beta T} \sin \omega T) \\ &\qquad - c_2(\omega + \omega e^{-\beta T} \cos \omega T - \beta e^{-\beta T} \sin \omega T) = - 2a \ . \end{aligned} \tag{28}$$

However, we have already commented that

$T \neq \frac{\pi}{\omega}$ (because $(2\beta - \zeta) \neq 0$). Then $\sin \omega T \neq 0$, and hence we can multiply both sides of this last equation by the term $e^{\beta T}\sin \omega T$, and simplify (using (27) to replace the term $c_2 \sin \omega T$) to yield:

$$c_1(2\beta e^{\beta T} \sin \omega T + \omega - \omega e^{2\beta T}) = - 2ae^{\beta T} \sin \omega T \ ,$$

or,

$$\sinh \beta T = \frac{(2a + \beta c_1)}{\omega c_1} \sin \omega T \quad ,$$

which, using (26), reduces to:

$$\sinh \beta T = - \frac{(\omega^2 + \beta\zeta - \beta^2)}{\omega\,(2\beta - \zeta)} \sin \omega T \quad . \tag{29}$$

Equation (29) expresses the necessary condition. In order to guarantee sufficiency, i.e., in order that a solution of (29) must generate a limit cycle, we must add the restriction,

$$T < T_o \quad ,$$

where T_o was defined above.

Applying the definition of T_o to the solutions given by (23) and (24) yields:

$$c_1 = - \frac{(2\beta - \zeta)}{(\beta^2 + \omega^2)}\, a \quad ,$$

$$c_1(e^{\beta T_o} - \cos \omega T_o) = c_2 \sin \omega T_o \quad ,$$

and,

$$-c_1 e^{-\beta T_o}(\beta \cos \omega T_o + \sin \omega T_o)$$

$$+ c_2 e^{-\beta T_o}(\omega \cos \omega T_o - \beta \sin \omega T_o) = 0 \quad .$$

Note that the first two equations are identical to (26) and (27). This follows since the conditions on $x_2(t)$ implied by both (22) and (25) are identical. Again we can assume $\sin \omega T_o \neq 0$, multiply both sides of the last equation by $e^{\beta T_o} \sin \omega T_o$, simplify by replacing the term $c_2 \sin \omega T_o$ by its value in terms of the middle equation, and divide by c_1, to yield;

$$\omega \cos \omega T_o - \beta \sin \omega T_o = \omega\, e^{-\beta T_o} \quad . \tag{30}$$

In view of our earlier comments then, we must pick the smallest positive solution of (30).

Mathematically, then we can summarize our results in the following:

Theorem: The system (20) will have a limit cycle of period $2T$ if and only if T satisfies:

$$\sinh \beta T = - \frac{\omega^2 + \beta\zeta - \beta^2}{\omega(2\beta - \zeta)} \sin \omega T, \tag{29}$$

and,

$$T < T_o ,$$

where:

T_o is the smallest positive solution of

$$\omega \cos \omega T_o - \beta \sin \omega T_o = \omega e^{-\beta T_o} . \tag{30}$$

C. Example - The Case Considered by Frederiksen [1]. To determine validity of the predictions of this model, consider the case used by Frederiksen [1]. Figure 6 shows the table of values appropriate to this case, which describes the F-94, a straight wing, single engine jet. The value of the coefficients appropriate to the linearized differential equations were computed and are shown in figure 7. Note that our assumptions on small size of certain coefficients seems valid. Using the values of β, ω and ζ so computed, equations (29) and (30) were graphically and numerically solved, and are graphically portrayed in figures 8 and 9, respectively. There are obviously two solutions of (29) in the region $\frac{\pi}{\omega} < T < \frac{2\pi}{\omega}$ (which is the region of interest since $(2\beta-\zeta) > 0$.) These are

$$T = 1.87 \qquad \text{and} \qquad T = 3.56$$

Parameters for F-94 [1]

$\rho = 2.38 \times 10^{-3}$

$\bar{c} = 6.40$

$m = 384$

$I_y = 2.65 \times 10^4$

$S = 239.0$

$C_{LO} = 1.440$

$C_{L\delta_e} = 0.43$

$C_{L\alpha_c} = 0.30$

$C_{L\dot{\theta}} = 10^{-5}$

$C_{L\Delta} = 0.05$

$C_{MO} = 0.15$

$C_{M\alpha} = 4.27$

$C_{M\alpha} = 1.54$

$C_{M\delta_e} = 0.88$

$C_{M\dot{\theta}} = 8.16$

$\theta_o = 0.099$ rad

$v_o = 169$ ft/sec

$\alpha_o = 0.41$ rad

$T = 0.0$

$\alpha_c = 0.41$ rad

[1] Values used by Frederiksen [1].

Figure 6

Based on the parameters in figure 6,

$$\frac{qS\bar{c}}{2mv_o^{\ 2}} C_{L\dot{\theta}} = 2.4 \times 10^{-8} , \quad \frac{\bar{c}}{2v_o} C_{M\dot{\alpha}} = 0.081$$

$$\frac{g}{v_o} \sin (\theta_o - \alpha_o) = - .058$$

$$\frac{g}{v_o} \sin (\theta_o - \alpha_o) - \frac{T}{mv_o} \cos \alpha_o - \frac{q^S}{mv_o} C_{L\alpha_c} = - .096$$

$$\frac{qS\bar{c}}{I_y} C_{M\alpha} = (\beta^2 + \omega^2) = 3.00$$

$$\frac{qS\bar{c}^2}{2v_o I_y} \{C_{M\dot{\theta}} + C_{M\dot{\alpha}}\} = 2\beta = 0.46, \text{ so } \beta = 0.23, \ \omega = 1.72$$

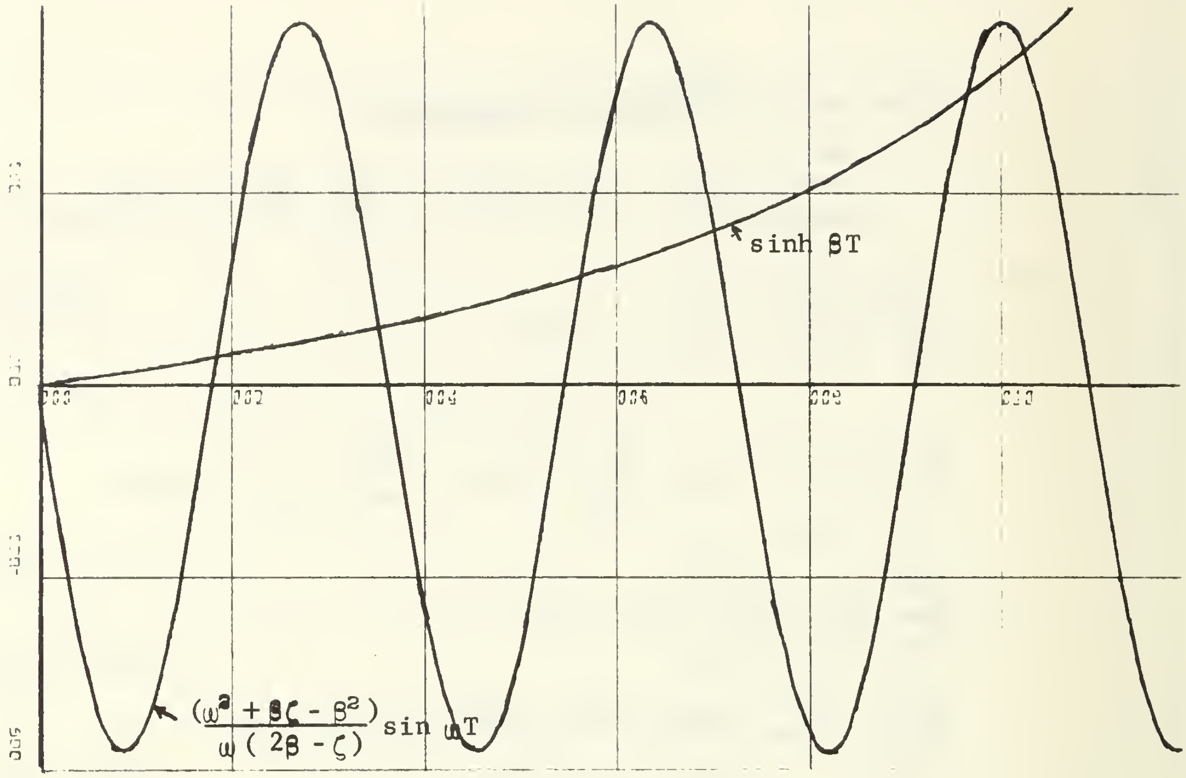
$$\frac{qS}{mv_o} C_{L\Delta} = a = 6.3 \times 10^{-3}$$

$$\frac{S\bar{c}^2}{2v_o I_y} C_{M\dot{\alpha}} = \zeta = 0.16$$

$$(2\beta - \zeta) = 0.30 > 0.$$

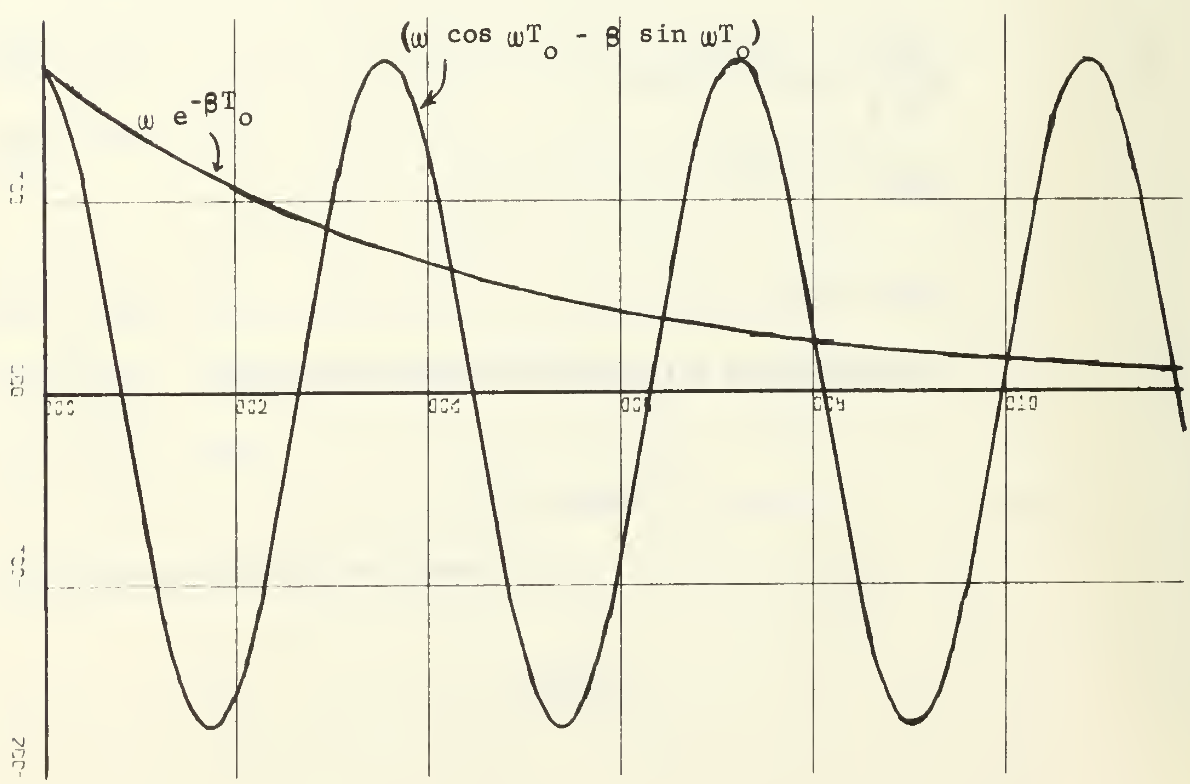
Coefficients for the Differential Equations

Figure 7



Graphical Solution of (29)

Figure 8



Graphical Solution of (30)

Figure 9

But T_o, defined as the first positive intersection of the curves in figure 9, is given by

$$T_o = 3.10$$

Thus, only the lower value of T satisfies the requirement $\frac{\pi}{\omega} < T < T_o$ and there will be exactly one limit cycle, with period

$$2T = 3.74 \text{ sec}$$

Computing c_1 and c_2 according to (26) and (27), and using the values to describe $x_2(t) = \Delta\alpha(t)$ as given by (24), leads to an amplitude for this limit cycle of

$$\Delta\alpha_{max} = .018 \text{ rad} \doteq 1^0$$

These values are in very good agreement with those obtained by Frederiksen, which were

$$2T = 3.77 \text{ sec}$$
$$\Delta\alpha_{max} = .020 \text{ rad} \quad .$$

D. Additional Comment. We shall make one other fairly important observation on the mathematical nature of (29) and (30). Note that (29) is equivalent to

$$\frac{\sinh \beta T}{\sin \omega T} = -\frac{\omega^2 + \beta\zeta - \beta^2}{\omega(2\beta-\zeta)} \tag{31}$$

Now since $(2\beta-\zeta) > 0$, the region of interest for this equation is $\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$. Letting

$$G(t) = \frac{\sinh \beta t}{\sin \omega t} \quad , \tag{32}$$

we have

$$G(t) < 0, \text{ in } \frac{\pi}{\omega} < T < \frac{2\pi}{\omega} \quad ,$$

and,

$$G(\frac{\pi}{\omega}^{+}) = G(\frac{2\pi}{\omega}^{-}) = -\infty$$

Furthermore

$$G'(t) = \frac{\beta \cosh \beta t \sin \omega t - \omega \sinh \beta t \cos \omega t}{\sin^2 \omega t}$$

Note the presence of the square in the denominator implies that the algebraic sign of $G'(t)$ is the same as the sign of the numerator.

This numerator,

$$\beta \cosh \beta t \sin \omega t - \omega \sin \beta t \cos \omega t$$

has the value

$$\omega \sinh \frac{\beta\pi}{\omega} > 0 \text{ at } t = \frac{\pi}{\omega} \quad ,$$

and the value

$$-\omega \sinh \frac{2\beta\pi}{\omega} < 0 \text{ at } t = \frac{2\pi}{\omega} \quad .$$

The derivative of this numerator is

$$(\beta^2 + \omega^2) \sinh \beta t \sin \omega t.$$

Since this derivative is negative on $\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$, then it clearly follows that the numerator of $G'(T)$ starts positive at $\frac{\pi}{\omega}$, remains positive up to some point T_m, $\frac{\pi}{\omega} < T_m < \frac{2\pi}{\omega}$, then turns negative, staying negative for the remainder of the interval. Based on earlier comments, $G'(t)$ behaves the same. Thus $G(t)$ is monotonically increasing on $\frac{\pi}{\omega} < T < T_m$, and monotonically decreasing on $T_m < T < \frac{2\pi}{\omega}$. Hence $G(t)$ has a unique maximum in $\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}$. [Figure 10 shows the general form of $G(t)$.]

But now let's investigate the relative positions of T_o and T_m . At T_o, $G'(T_o)$ has the numerator

$$\beta \cosh \beta T_o \sin \omega T_o - \omega \sinh \beta T_o \cos \omega T_o$$

$$= \frac{1}{2} e^{\beta T_o} [\beta \sin \omega T_o - \omega \cos \omega T_o] + \frac{1}{2} e^{-\beta T_o} [\beta \sin \omega T_o + \omega \cos \omega T_o].$$

But according to (30),

$$\beta \sin \omega T_o = - \omega e^{-\beta T_o} + \omega \cos \omega T_o$$

Thus, the numerator of $G'(T_o)$ is

$$= \frac{1}{2} e^{\beta T_o} [-\omega e^{-\beta T_o}] + \frac{1}{2} e^{-\beta T_o} [-\omega e^{- T_o} + 2\omega \cos \omega T_o]$$

$$= \frac{\omega}{2} e^{-\beta T_o} [\cos \omega T_o - \cosh \omega T_o] < 0 \quad .$$

Then

$G'(T_o) < 0$, i.e., $G(T)$ is decreasing at T_o , and so by our earlier comment, $T_m < T_o$.

But observe that the right hand side of (31) is independent of T . Thus, since (29) and (31) are equivalent, a necessary and sufficient condition for (29) to have a solution, based on our analysis of $G(T)$, is

$$\max_{\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}} \left[\frac{\sinh \beta t}{\sin \omega t}\right] \geq \frac{\omega^2 + \beta\zeta - \beta^2}{\omega(2\beta-\zeta)} \quad . \tag{33}$$

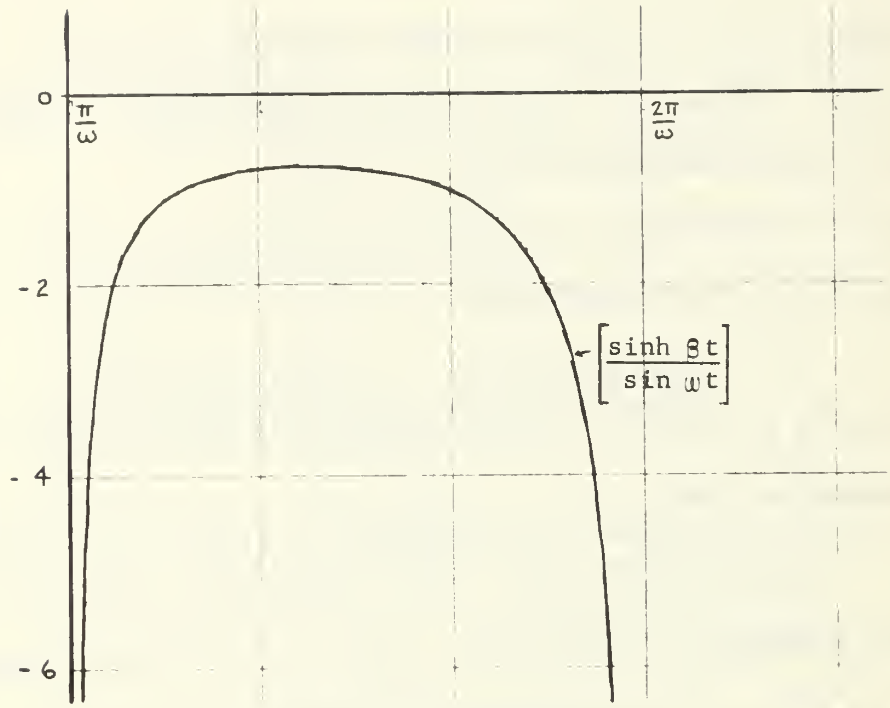


Figure 10

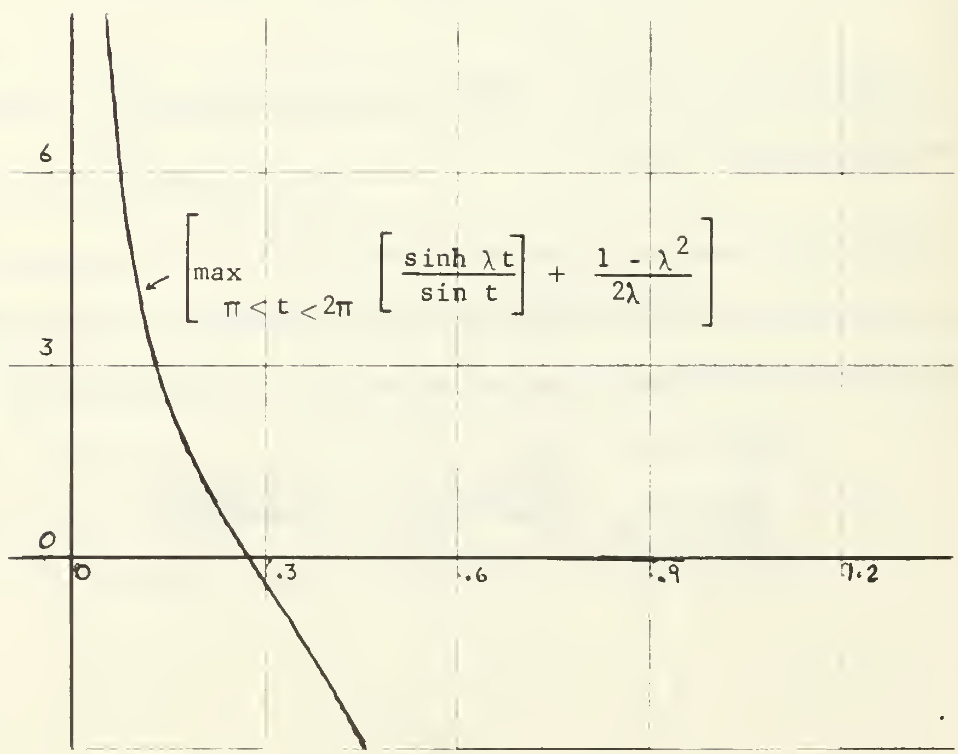


Figure 11

Finally, since $G(\frac{\pi}{\omega}^+) = -\infty$ and $T_m < T_o$, then if (33) is satisfied, then there must be a $T < T_m < T_o$ such that

$$\frac{\sinh \beta T}{\sin \omega T} = - \frac{\omega^2 + \beta\zeta - \beta^2}{\omega(2\beta-\zeta)} .$$

Then this T satisfies (29) and (30), and so generates a limit cycle. Thus, (33) express a necessary and sufficient condition for existence of a limit cycle in this model.

This is expressed by the following:

Theorem: The system (20), where $(2\beta-\zeta) > 0$, will have a limit cycle solution if and only if

$$\max_{\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}} \left[\frac{\sinh \beta t}{\sin \omega t}\right] \geq - \frac{\omega^2 + \beta\zeta - \beta^2}{\omega(2\beta-\zeta)}$$

Observe that, with the substitution

$$\lambda = \frac{\beta}{\omega} > 0 , \quad \text{and} \quad \eta = \frac{\zeta}{\omega} > 0 ,$$

the necessary and sufficient condition becomes

$$\max_{\pi < t < 2\pi} \left[\frac{\sinh \lambda t}{\sin t}\right] \geq \frac{1 + \eta\lambda - \lambda^2}{(2\lambda - \eta)} , \tag{34}$$

(the left hand side of which is clearly a function of λ only.) Note we already have commented that $(2\beta - \zeta) > 0$, hence

$$(2\lambda - \eta) > 0, \quad \text{or} \quad 0 < \eta < 2\lambda .$$

Now, observe that

$$\frac{d}{d\eta}\left[-\frac{1+\eta\lambda-\lambda^2}{(2\lambda-\eta)}\right] = -\frac{\lambda^2+1}{(2\lambda-\eta)^2} < 0$$

and therefore the right-hand side of (34) has its maximum value at $\zeta = 0$. Thus, clearly, a sufficient condition for (34) to be satisfied, and hence for a limit cycle to exist, is,

$$\max_{\pi<t<2\pi}\left[\frac{\sinh \lambda t}{\sin t}\right] \geq \frac{\lambda^2-1}{2\lambda} . \tag{35}$$

In figure (11) we plot the function

$$\max_{\pi<t<2\pi}\left[\frac{\sinh \lambda t}{\sin t}\right] + \frac{1-\lambda^2}{2\lambda}$$

for $0 < \lambda < 1$, and observe that (35) cannot be satisfied if

$$\lambda > .280 .$$

(Observe (35) cannot be satisfied for $\lambda \geq 1$.) Thus, a sufficient condition for existence of a limit cycle is:

$$\lambda \leq .280 \tag{36}$$

Now, if $\lambda \leq .280$, and $\beta = \lambda\omega$, we have:

$$\omega^2 \leq \omega^2 + \beta^2 = (1+\lambda^2)\,\omega^2 \leq 1.078\,\omega^2 ,$$

and so we can approximate, $\omega \doteq \sqrt{\omega^2+\beta^2}$.

But then, according to the definitions (17), $\lambda = \frac{\beta}{\omega}$ is proportional to

$$\frac{C_{M\dot{\alpha}} + C_{M\dot{\theta}}}{C_{M\alpha}} ,$$

and we see our sufficient condition (35) translates into a statement that

$$C_{M\dot{\alpha}} + C_{M\dot{\theta}}$$

is somehow small compared to $C_{M\alpha}$.

Since $C_{M\alpha}$ is analogous to a spring constant, and $C_{M\dot{\theta}}$ and $C_{M\dot{\alpha}}$ are damping terms, we can view (35) and (36) as saying:

"A sufficient condition for existence of a limit cycle behavior in the presence of a sharp negative lift slope, is that the damping terms in C_M are small relative to the "restoring" term, $C_{M\alpha}$."

Clearly this interpretation makes physical "sense".

4. SUMMARY AND CONCLUSIONS

In this paper, we have developed and analyzed a simple two-dimensional model for post-stall time histories in the presence of a sharp lift loss at stall. This model yields a simple necessary and sufficient condition for the occurrence of limit cycle, i.e., "rocking chair" stall. Furthermore, the model predicts values that are in very close agreement with those obtained from numerical solution of the full, four-dimensional, non-linear case. Lastly, the model yields physically satisfying information about the important parameters for occurrence of this limit cycle.

APPENDIX. Definition of Variables and Coefficients

$\bar{c}$ = Mean Aerodynamic Chord (MAC)

C_D = Drag Coefficient

C_L = Lift Coefficient

C_M = Pitching Moment

g = Gravitation Acceleration

I_y = Moment of Inertia about c.g.

m = Mass

q = $\frac{1}{2} \rho v^2$

S = Wing Reference Area

T = Thrust

u = Velocity Component along Thrust Line

v = Velocity

w = Velocity Compenent normal to Thrust Line

α = Angle of Attack

θ = Pitch Angle

ρ = Density of Air

REFERENCE

1. Frederiksen, John T., An Evaluation of the Longitudinal Dynamic Stability of an Aircraft at Stall, M. S. Thesis, Naval Postgraduate School, Monterey, CA, June 1972

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A simplified non-linear relay model is developed to describe observed post-stall oscillations in aircraft. The predictions of the model are evaluated against results obtained by numerical techniques, and shown to yield close agreement.